INJECTORS AND NORMAL SUBGROUPS OF FINITE GROUPS

BY
AVINOAM MANN

ABSTRACT

A class of subgroups, $\mathcal{N}$-injectors, previously defined only for soluble groups, is here defined for some non-soluble groups. It is proved that injectors have some properties which resemble those of Sylow subgroups.

In this note, all groups are finite. Let G be a finite group. A subgroup $S \subseteq G$ is an *$\mathcal{N}$-injector* of G if, given any subnormal subgroup H of G, $S \cap H$ is a maximal nilpotent subgroup of H. This concept is due to B. Fischer, who proved that, if G is soluble, then $\mathcal{N}$-injectors exist and any two of them are conjugate [1]. The object of the present paper is to show that this concept can be applied also to non-soluble groups. Thus, we first reprove Fischer's results for groups G satisfying the condition (weaker than solubility) $C_G(F(G)) \subseteq F(G)$, where $F(G)$ denotes the Fitting subgroup of G. We then show that in such groups the $\mathcal{N}$-injectors have some properties similar to those of Sylow subgroups. In particular, we give analogues of the well-known fusion theorem of Alperin and "ZJ theorem" of Glauberman.

1. *$\mathcal{N}$-injectors. Definition.* A group G is *$\mathcal{N}$-constrained, if* $C_G(F(G)) \subseteq F(G)$. Let $S = S(G)$ denote the maximal normal solvable subgroup of G.

PROPOSITION 1. *G is $\mathcal{N}$-constrained if and only if* $C_G(S(G)) \subseteq S(G)$.

PROOF. Let G be $\mathcal{N}$-constrained. Then $C_G(S(G)) \subseteq C_G(F(G)) \subseteq F(G) \subseteq S(G)$. Conversely, assume that $C_G(S) \subseteq S(G)$. Denote $C = C_G(F(G))$. Then $C \lhd G$, $C \cap S = Z(F(G))$, so C stabilizes the chain $S \supseteq Z(F(G)) \supseteq \langle 1 \rangle$, so C induces on S an abelian group of automorphisms, $C/C_C(S)$ is abelian, and $C_C(S) \subseteq S$ by assumption, thus C is soluble, and $C \subseteq S$. Since S is soluble and $F(G) = F(S)$, we get $C = C_S(F(S)) \subseteq F(S)$.

Received August 6, 1970 and in revised form January 10, 1971

THEOREM 1. *Let G be an $\mathcal{N}$-constrained group. Then,*

a) *G contains $\mathcal{N}$-injectors.*

b) *Any two $\mathcal{N}$-injectors of G are conjugate.*

c) *A subgroup S of G is an $\mathcal{N}$-injector if and only if S is a maximal nilpotent subgroup of G containing the Fitting subgroup.*

PROOF. Let $F = F(G)$ be the Fitting subgroup of G. For a prime p let $F_{p'}$ be the p-complement of F, and denote $C_p = C_G(F_{p'})$. Let S_p be a Sylow p-subgroup of C_p. For the purposes of this proof only, we define a *pre-injector* of G to be the subgroup $S = \langle S_p \mid p$ runs over the prime divisors of $|F(G)|\rangle$. Different choices of the S_p's yield different pre-injectors. We claim

(*) *If $p \neq q$, then C_p normalizes S_q, and S_p centralizes S_q.*

Indeed, since $C_p \lhd G$ and $C_q \lhd G$, we have

$$[C_{p'}, C_q] \subseteq C_p \cap C_q = C_G(F(G)) \subseteq F(G).$$

In particular, $[C_{p'}, S_q] \subseteq F$, so C_p normalizes $S_q F = S_q \times F_{q'}$, and therefore C_p normalizes S_q. As S_p and S_q normalize each other and have co-prime orders, they centralize each other. In particular, we see that $S = \Pi S_p$ is a nilpotent subgroup.

Let $S \subseteq T$, T nilpotent. We have $F \subseteq S$, therefore T_p, the Sylow p-subgroup of T, centralizes $F_{p'}$, $T_p \subseteq C_p$, and as S_p is a Sylow p-subgroup of C_p, $T_p = S_p$ and $T = S$. On the other hand, if $T \supseteq F$ is an arbitrary nilpotent subgroup, we still get $T_p \subseteq C_p$, and thus there exists an S_p for which $T_p \subseteq S_p$, i.e. there exists a pre-injector S such that $S \supseteq T$. Thus, the pre-injectors of G coincide with the maximal nilpotent subgroups of G containing F.

Let $S_1 = \Pi(S_1)_p$ be another pre-injector. For each p, let $a_p \in C_p$ be such that $(S_1)_p = S_p^{a_p}$. If $q \neq p$, then by (*), a_p normalizes S_q and $(S_1)_q$. Therefore $S_1 = S^a$, where $a = \Pi a_p$ (in any order), so any two pre-injectors of G are conjugate and b. and c. of Theorem 1 hold for pre-injectors.

Let $H \lhd G$. Then $F(H) = F(G) \cap H$. Let $D = C_H(F(H))$. Then $[D, F] \subseteq F \cap H$, so D stabilizes the chain $F \supseteq F \cap H \supseteq \{1\}$, so $D/C_D(F)$ is nilpotent. However, $C_D(F) \subseteq D \cap F = D \cap F(H) \subseteq Z(D)$, therefore D is nilpotent, and as $D \lhd H$, we get $D \subseteq F(H)$ and H is $\mathcal{N}$-constrained.

Let $T = \Pi T_p$ be a pre-injector of H. Since $F(H)_{p'} = F_{p'} \cap H$, we see that T_p stabilizes the chain $F_{p'} \supseteq F_{p'} \cap H \supseteq \{1\}$, so T_p, a p-group, induces on $F_{p'}$ a p'-group of automorphisms, so T_p centralizes $F_{p'}$, and $T_p \subseteq S_p$ for an approriate S_p. However, S_p centralizes $F_{p'}$; hence, certainly $F(H)_{p'}$, so $S_p \cap H \subseteq T_p$ and $T = S \cap H$. By repeating the argument, it follows that if $H \lhd\lhd G$, then H is $\mathcal{N}$-constrained and $S \cap H$ is a pre-injector of H. In particular, $S \cap H$ is a maximal nilpotent subgroup of H, so S is an $\mathcal{N}$-injector of G and the theorem is proved.

We next note some further properties of $\mathcal{N}$-injectors. It should be mentioned that theorems 1 and 2 were proved by Fischer for G a soluble group.

THEOREM 2. *Let G be $\mathcal{N}$-constrained. Let $F = F(G)$, S be an $\mathcal{N}$-injector of G and H be a subgroup containing F.*

a) *H is $\mathcal{N}$-constrained. Denote by T an $\mathcal{N}$-injector of H.*

b) *$T = S_1 \cap H$, for some $\mathcal{N}$-injector S_1 of G.*

c) *$S \cap H$ is contained in some $\mathcal{N}$-injector of H.*

d) *If $H \subseteq K$, then T is contained in an $\mathcal{N}$-injector of K.*

e) *If $S \subseteq H$, then S is an $\mathcal{N}$-injector of H.*

PROOF. We certainly have $F \subseteq F(H)$, so $C_H(F(H)) \subseteq C_G(F) \subseteq F \subseteq F(H)$, and H is $\mathcal{N}$-constrained.

Now, T is a nilpotent subgroup of G containing F, so by Theorem 1.c $T \subseteq S_1$, for S_1 an $\mathcal{N}$-injector of G. Then $T = S_1 \cap H$, because T is a maximal nilpotent subgroup of H.

Let $p \neq q$ be primes, S_p—a Sylow p-subgroup of S, $(S_1)_{q'}, F(H)_q$—the Sylow q-subgroups of S_1 and $F(H)$. Then S_p centralizes $(S_1)_q$, hence $F(H)_q$, by (*). Therefore, S_p centralizes $F(H)_{p'}$, the p-complement of $F(H)$, so $S_p \cap H$ is contained in some Sylow p-subgroup of $C_H(F(H)_{p'})$, so, according to the construction of $\mathcal{N}$-injectors given in the proof of Theorem 1, $S \cap H \subseteq T_1$ for some $\mathcal{N}$-injector T_1 of H. This proves c.

For d. write $T = S_1 \cap H$ as in b. then $T \subseteq S_1 \cap K$ and $S_1 \cap K$ is contained in an $\mathcal{N}$-injector of K by c.

Finally, e. follows from c. and the maximality from S.

We now have enough properties of $\mathcal{N}$-injectors to prove Alperin's fusion theorem, which we formulate as folllows:

THEOREM 3. *Let G be an $\mathcal{N}$-constrained group, S—an $\mathcal{N}$-injector of G, A and B subsets of S, and $A = B^x$, $x \in G$. Then, $x = x_1 \cdots x_n$, where each x_i is in the normalizer of some subgroup of S that contains $A^{x_1 \cdots x_{i-1}}$.*

The stronger form of this result, as given e.g. in [4, Theorem 7.2.6] also holds ("p-element of a group H" should be changed to "an element of some $\mathcal{N}$-injector of H"). The proof is identical to that of [4], where the properties of $\mathcal{N}$-injectors listed in Theorem 2 replace familar properties of Sylow subgroups. We note that these properties are needed for subgroups of the type $H = N_G(S \cap T)$, where S and T are $\mathcal{N}$-injectors, so $H \supseteq S \cap T \supseteq F(G)$, and Theorem 2 applies.

2. The Thompson subgroup

DEFINITION. A group G is *$\mathcal{N}$-stable* if, given any nilpotent subnormal subgroup N, and an element x such that $x \in N_G(N)$ and $[N, x, x] = 1$, we have $x \in F(N_G(N) \bmod C_G(N))$.

PROPOSITION 2. *Suppose that $SL(2, p)$ is not involved in G, for all primes p satisfying $p \mid |F(G)|$. Then G is $\mathcal{N}$-stable.*

PROOF. Let N and x be as in the definition. Then $\langle N, x\rangle$ is a nilpotent group. Let $p_1, \cdots, p_n$ be the primes dividing $|\langle N, x\rangle|$, and write $x = x_1 \cdots x_n$, where x_i is a p_i-element in $\langle N, x\rangle$. If $p_i \nmid |N|$, then $x_i \in C(N)$. If $p_i \mid |N|$, then $SL(2, p_i)$ is not involved in G by assumption, so $N_G(N)$ is p_i-stable in the sense of Glauberman [2, Def. 2.1, Lemma 6.3]. Therefore $x_i \in O_{p_i}(N_G(N_{p_i}) \bmod C(N_{p_i}))$, where N_{p_i} is a p_i-Sylow subgroup of N, while, if $q \neq p_i$, $x_i \in C(N_q)$, and so $x_i \in O_{p_i}(N_G(N) \bmod C_G(N)) \subseteq F(N_G(N) \bmod C_G(N))$, and $x \in F(N_G(N) \bmod C_G(N))$.

DEFINITION Let H be a group. Then $\mathscr{A}(H) = \{A \subseteq H \mid A$ is abelian and if $B \subseteq H$, B abelian, then $|A| \geqq |B|\}$.

$$J(H) = \langle A \mid \mathscr{A}(H)\rangle.$$

THEOREM 4. *Let G be $\mathcal{N}$-stable and $\mathcal{N}$-constrained group, with $|F(G)|$ odd. Let S be an $\mathcal{N}$-injector of G. Then $Z(J(S)) \triangleleft G$.*

PROOF. (This proof was suggested by the referee. Our original proof was modelled after Glauberman's proof of his ZJ-theorem, and was longer.) Let $F = F(G)$, let p be a prime, $p \mid |F(G)|$, let $C_p = C_G(F_{p'})$, and let S_p be a Sylow p-subgroup of S. Then $C_p \triangleleft G$, and S_p is a Sylow p-subgroup of C_p (by proof of Theorem 1). Now $O_{p'}(C_p)$ centralizes $F_p = O_p(C_p)$ as well as $F_{p'}$, so $O_{p'}(C_p) \subseteq F$, and therefore $O_{p'}(C_p) = Z(F_{p'})$. Let $\bar{C}_p = C_p / Z(F_{p'})$, then $O_{p'}(\bar{C}_p) = 1$. Now $Z(F_{p'}) \subseteq Z(C_p)$, so $F(C_p \bmod Z(F_{p'}))$ is a nilpotent (and normal) subgroup of G, so in particular $O_p(\bar{C}_p) = F_p Z(F_{p'}) / Z(F_{p'})$. Thus a normal p-subgroup of $\bar{C}_p$ is the image of a subgroup of $F_{p'}$, so $\mathcal{N}$-stability of G implies p-stability of $\bar{C}_p$. Also $C_{C_p}(F_p) = C_G(F_{p'}) \cap C_G(F_p) = C_G(F)$, so $\mathcal{N}$-constraint of G implies p-constraint of $\bar{C}_p$. Thus, by Glauberman's theorem, $Z(J(\bar{S}_p)) \lhd \bar{C}_p$ ($\bar{S}_p = S_p Z(F_{p'})/Z(F_{p'})$, and $Z(J(S_p)) \lhd C_p$. Now automorphisms of C_p transform S_p to subgroups conjugate to it, so $Z(J(S_p))$ char C_p and $Z(J(S_p)) \lhd G$. Letting p vary over all primes, we get $Z(J(S)) \lhd G$.

PROPOSITION 3. *Under the same assumptions as in Theorem* 4,

$$G = N(J(S))C(Z(J(S))) = N(J(S))C(Z(S)).$$

PROOF. Let $Z = Z(J(S))$, $C = C(Z)$. Since $Z \lhd G$, also $C \lhd G$, so $S \cap C$ is an $\mathscr{N}$-injector of C. Since $J(S) \subseteq S \cap C$, $J(S) = J(S \cap C)$, so by the Frattini argument, $G = N(J(S))C$. Since $Z(S) \subseteq Z(J(S))$, we have $C(Z(S)) \supseteq C$ and $G = N(J(S))CZ(S))$.

THEOREM 5. *Let G be soluble. Assume that $F(G)$ is not divisible by* 2 *or* 3; *if* $5 \mid |F(G)|$, *assume also that the Sylow* 2*-subgroup of G is abelian. Let S be an $\mathscr{N}$-injector of G. Then $J(S) \lhd G$.*

The proof is identical to that of Theorem 4, using, instead of the ZJ-theorem, the following result of Thompson [3, p. 164]: if G is soluble, $p > 5$, or $p = 5$ and a Sylow 2-subgroup of G is abelian, and if $O_{p'}(G) = 1$, then $J(G_p) \lhd G$.

To illustrate the possible use of our results, consider the following situation. Let G be any finite group. Recall that a *local* subgroup is one which is the normalizer of a non-identity nilpotent subgroup. Let M be a maximal local subgroup of G. Assume that M is $\mathscr{N}$-constrained and $\mathscr{N}$-stable, and let S be an $\mathscr{N}$-injector of M. Then $Z = Z(J(S)) \lhd M$, so $M = N_G(Z)$ by maximality of M. Let $S \subseteq T$, T nilpotent, then Z char S implies $N_T(S) \subseteq N_G(Z) = M$. But S is maximal nilpotent in M, so $N_T(S) = S$, which means that $T = S$. Thus, S is a maximal nilpotent subgroup of F.

REFERENCES

1. B. Fischer, W. Gaschütz and B. Hartley, *Injektoren Endlicher Auflösbarer Gruppen*, Math. Z. **102** (1967), 337–339.

2. G. Glauberman, *A characteristic subgroup of a p-stable group*, Canad. J. Math. **20** (1968), 1101–1135.

3. G. Glauberman, *Prime-power factor groups of finite groups*, Math. Z. **107** (1968), 159–172.

4. D. Gorenstein, *Finite groups*, New York, Harper and Row, 1968.

THE HEBREW UNIVERSITY OF JERUSALEM